Spontaneous staggered polarisations of the cyclic solid-on-solid models

1990 J. Phys. A: Math. Gen. 23 L163

(http://iopscience.iop.org/0305-4470/23/4/007)

J. Phys. A: Math. Gen. 23 (1990) L163-L167. Printed in the UK

LETTER TO THE EDITOR

Spontaneous staggered polarisations of the cyclic solid-on-solid models

Etsuro Date†, Michio Jimbo‡, Atsuo Kuniba§ and Tetsuji Miwa||

† Department of Mathematics, College of General Education, Kyoto University, Kyoto 606, Japan
‡ Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606, Japan
§ Department of Mathematics, Faculty of Science, Kyushu University, Fukuoka 812, Japan
|| Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606, Japan

Received 22 November 1989

Abstract. We compute the staggered polarisations of cyclic solid-on-solid models using the Bethe-type eigenvectors. Our derivation assumes some mathematical identities of elliptic functions which we have not yet proved.

Lattice models in statistical mechanics are called 'solvable' when the Boltzmann weights obey the Yang-Baxter equation (Baxter 1982). In these models, however, physical quantities of interest have not been 'solved' entirely. To our present knowledge there are mainly two quantities for which exact results can be derived by certain algorithms:

(i) free energy;

(ii) one-point function of the IRF (interactions-round-a-face) models.

In the first case the problem reduces to obtaining the largest eigenvalue of the row-to-row transfer matrix; in the second case the full set of eigenvalues of the corner transfer matrix (CTM) is needed (Baxter 1982). The n-point functions are more difficult since they require the knowledge of eigenvectors. These vectors can be described by the Bethe ansatz method, but they are quite complicated. We note some recent progress in this direction (Kirillov and Smirnov 1988, Its *et al* 1989, and references therein).

The spontaneous staggered polarisation (SSP) of the six-vertex model in the anti-ferroelectric regime was obtained by Baxter (1973a). This is the one-point function of a bond in the vertex model language, and also the two-point function of the nearest-neighbour sites in the IRF model language. A simplifying feature here is that this quantity can be expressed by using only the eigenvectors corresponding to the largest eigenvalues.

In this letter we treat analogous quantities for the cyclic solid-on-solid (CSOS) models (Kuniba and Yajima 1988, Pearce and Seaton 1988). These are IRF models having elliptic function parametrisation. The corresponding Bethe-type eigenvectors have been constructed earlier (Baxter 1973b). Following closely Baxter's analysis of the six-vertex model, we find an expression for SSP (in regime III of Kuniba and Yajima 1988) in terms of elliptic theta functions. In deriving them we need two mathematical identities ((A), (B) below) which we have not proved yet, so our final result is still a conjecture. The working will be given in a subsequent article.

Let us recall the CSOS model. We consider a two-dimensional square lattice of size $(M+1)\times N$ with M, N even. Fix an integer $L\geq 3$. With each site (i,j) $(0\leq i\leq M,$

0305-4470/90/040163+05$03.50

$1 \leq j \leq N$) we associate a fluctuation variable l_{ij} which takes values in Z/LZ. In the horizontal direction we impose the cyclic boundary condition $l_{i N+1} = l_{i1}$. Along the top and bottom rows we impose one of the $2(L-1)$ boundary conditions $B^{(s)}$ $(0 \leq s \leq 2(L-1)-1)$,

$$B^{(0)}: \quad l_{0,2j} = l_{M,2j} = 0$$

$$B^{(s)}(1 \leq s \leq L-2): \quad l_{0,2j+1} = l_{M,2j+1} = s+1,\ l_{0,2j} = l_{M,2j} = s$$

$$B^{(L-1)}: \quad l_{0,2j+1} = l_{M,2j+1} = 0$$

$$B^{(s)}(L \leq s \leq 2L-3): \quad l_{0,2j+1} = l_{M,2j+1} = s-L+1,\ l_{0,2j} = l_{M,2j} = s-L+2.$$

The partition function is

$$Z_{MN}(s) = \sum_l{}^{(s)} \mathrm{e}^{-\beta E(l)} \qquad \mathrm{e}^{-\beta E(l)} = \prod_{\substack{1 \leq i \leq M \\ 1 \leq j \leq N}} {}^{l_{i-1j}}_{l_{ij}}\boxed{w_{ij}}^{l_{i-1j+1}}_{l_{ij+1}}.$$

The sum $\Sigma_l^{(s)}$ is over all values of $l = (l_{ij})_{0 \leq i \leq M, 1 \leq j \leq N}$ satisfying the cyclic solid-on-solid condition $l_{i+1j} - l_{ij} = l_{ij+1} - l_{ij} \equiv \pm 1 \bmod L$ and the boundary condition $B^{(s)}$. The Boltzmann weights are given as follows:

$${}^{l\pm1}_{l}\boxed{w}^{l}_{l\mp1} = \frac{[\frac{1}{2}+w]([l+1]_4[l-1]_4)^{1/4}}{[1][l]_4^{1/2}}$$

$${}^{l}_{l\mp1}\boxed{w}^{l\pm1}_{l} = \frac{[\frac{1}{2}-w]([l+1]_4[l-1]_4)^{1/4}}{[1][l]_4^{1/2}}$$

$${}^{l}_{l\pm1}\boxed{w}^{l\pm1}_{l} = \frac{[l \mp w \pm \frac{1}{2}]_4}{([l]_4[l\pm1]_4)^{1/2}}.$$

Here $[u]$, $[u]_4$ denote the elliptic theta functions; $[u] = \theta_1(\pi u/L, p)$, $[u]_4 = \theta_4(\pi u/L, p)$. The choice of the spectral parameters $w_{ij} = w_j - w'_i$ makes the model Z-invariant in the sense of Baxter (1978).

Our aim is to compute the spontaneous staggered polarisations $P(s)$ of this model in the thermodynamic limit, i.e.

$$P(s) = \lim_{M,N\to\infty} P_{MN}(s)$$

$$P_{MN}(s) = \sum_l{}^{(s)} \frac{\mathrm{e}^{-\beta E(l)}}{Z_{MN}(s)} (\delta^{(L)}(l_{M/2\,1}, l_{M/2\,N}+1) - \delta^{(L)}(l_{M/2\,1}, l_{M/2\,N}-1))$$

where

$$\delta^{(L)}(l, l') = \begin{cases} 1 & \text{if } l - l' \equiv 0 \bmod L \\ 0 & \text{otherwise.} \end{cases}$$

We treat the case $0 < p < 1$. The choice of w_{ij} will be specified below. Following Baxter (1973b) we define the transfer matrix V_i $(1 \leq i \leq M)$ of this model:

$$V_i|l_1, \ldots, l_N\rangle = \sum_m \prod_{j=1}^{N} {}^{l_j}_{m_j}\boxed{w_{ij}}^{l_{j+1}}_{m_{j+1}} |m_1, \ldots, m_N\rangle.$$

Set

$$|s, 0\rangle = \sum_l{}^{(s)} |l_1, \ldots, l_N\rangle. \tag{1}$$

The sum is over $l=(l_j)_{1\leq j\leq N}$ satisfying $l_j=l_{0j}$ where $(l_{0j})_{1\leq j\leq N}$ is restricted by the boundary condition $B^{(s)}$. (Except in the cases $s=0$ or $s=L-1$, the right-hand side comprises only one term.) We define dual vectors $\langle l_1,\ldots,l_N|$, $\langle s,0|$, similarly. Let σ_N be the diagonal operator such that

$$\sigma_N|l_1,\ldots,l_N\rangle=\begin{cases}|l_1,\ldots,l_N\rangle & \text{if } l_1=l_N+1\\ -|l_1,\ldots,l_N\rangle & \text{if } l_1=l_N-1.\end{cases}$$

Then we have

$$P_{MN}(s)=\frac{\langle s,0|V_M\ldots V_{M/2+1}\sigma_N V_{M/2}\ldots V_1|s,0\rangle}{\langle s,0|V_M\ldots V_1|s,0\rangle}. \tag{2}$$

We will rewrite (2) in terms of eigenvectors of the commuting transfer matrices $V_1,\ldots,V_M$. Let $|f\rangle=\Sigma_l f(l_1,\ldots,l_N)|l_1,\ldots,l_N\rangle$ be the eigenvector constructed by the Bethe ansatz method (Baxter 1973b). It is given as follows. Fix n such that $N-2n\equiv 0 \bmod L$. For our purpose it is sufficient to consider the case $n=N/2$. For $(l_1,\ldots,l_N)$ satisfying $l_{j+1}=l_j\pm 1$ $(l_{N+1}=l_1)$, we define a set of integers $(J_1,\ldots,J_n)$, $J_1<\ldots<J_n$, so that $l_{J_a+1}=l_{J_a}-1$ $(a=1,\ldots,n)$ and $l_{j+1}=l_j+1$ $(j\neq J_1,\ldots,J_n)$. Then we have

$$f(l_1,\ldots,l_N)=\sum_P\prod_{a<b}\frac{[u_{P_a}-u_{P_b}+1]}{[u_{P_a}-u_{P_b}]}\prod_{a=1}^{n} g_{P_a}(l-2a+2,J_a). \tag{3}$$

The sum is over the permutations of $\{1,\ldots,n\}$. The single particle function $g_a(l,J)$ is given by

$$g_a(l,J)=\left(\prod_{j=1}^{J-1}\frac{[\frac{1}{2}+w_j+u_a]}{[\frac{1}{2}-w_j-u_a]}\right)\frac{[1][l+J-\frac{3}{2}-w_J-u_a]_4}{[\frac{1}{2}-w_J-u_a]([l+J-1]_4[l+J-2]_4)^{1/2}}.$$

The parameters $u_1,\ldots,u_n$ must solve the Bethe ansatz equation,

$$\prod_{j=1}^{N}\frac{[\frac{1}{2}+w_j+u_a]}{[\frac{1}{2}-w_j-u_a]}=(-1)^{n+1}\prod_{b=1}^{n}\frac{[1+u_a-u_b]}{[1-u_a+u_b]}.$$

The left eigenvector $\langle f|$ is given by

$$\langle f|=\sum_l f^*(l_1,\ldots,l_N)\langle l_1,\ldots,l_N|$$

where $f^*(l_1,\ldots,l_N)$ is given by (3) with w_j and u_a replaced by $-w_j$ and $-u_a$.

We assume that $P(s)$ is independent of the choice of w_j, w'_i as long as $|\mathrm{Re}\, w_{ij}|<C$ for some positive constant C (Baxter 1978). In the following, we choose the following special values (Baxter 1973a):

$$w'_i=0\ (i=1,\ldots,M) \qquad w_j=\frac{L\tau}{2N}(2j-N-1)\ (j=1,\ldots,N) \tag{4}$$

where τ is related to p via $p=e^{2\pi i\tau}$. Let $x=e^{-\pi i/L\tau}$, and introduce new variables $z_a=x^{2u_a}$, $\zeta=x^{2(u_1+\ldots+u_n)/L}$. Then for each k $(0\leq k\leq 2(L-1)-1)$ the Bethe ansatz equation has a solution that is uniquely determined by the condition $z_a^{(k)}|_{x=0}=\omega_{2n(L-1)}^{Lk+(L-1)(n-2a-1)}$, $\zeta^{(k)}|_{x=0}=\omega_{2n(L-1)}^{nk}$. Here and in what follows $\omega_m=e^{2\pi i/m}$. These solutions are of the form $u_a=\alpha(x)+a\eta$ $(\eta=L\tau/n, a=1,\ldots,n)$ and the corresponding eigenvalues constitute the largest band in the low-temperature phase (Pearce and Batchelor 1989). We denote by $f^{(k)}$ the function (3) determined by $(z_1^{(k)},\ldots,z_n^{(k)},\zeta^{(k)})$.

Now we define the Fourier-transformed vectors

$$|s\rangle = \frac{1}{\sqrt{2(L-1)}} \sum_{k=0}^{2(L-1)-1} \omega_{2(L-1)}^{k(s-s_0)} |f^{(k)}\rangle$$

$$\langle s| = \frac{1}{\sqrt{2(L-1)}} \sum_{k=0}^{2(L-1)-1} \omega_{2(L-1)}^{-k(s-s_0)} \langle f^{(k)}|$$

where $s_0 = 0$ (respectively $L-1$) if n is odd (respectively even). Then the expansion of $|s\rangle$ in terms of x reads as $|s\rangle = C_s x^{[(1-L)/2L]n^2}(|s, 0\rangle + x|s, 1\rangle + x^2|s, 2\rangle + \ldots)$ where $|s, 0\rangle$ is given by (1) and C_s is some constant.

We make the assumption that for each r the coefficient $|s, r\rangle$ is a linear combination of vectors $|l_1, \ldots, l_N\rangle$ such that $l_1, \ldots, l_N$ breaks the condition $B^{(s)}$ at most r times (cf section 3.1 of Baxter 1973a). This assumption and (2) lead us to the following expression of $P(s)$:

$$P(s) = \lim_{N\to\infty} \frac{\langle s|\sigma_N|s\rangle}{\langle s|s\rangle}. \tag{5}$$

To check the validity of the above assumption we have computed the local height probabilities using the formula similar to (5) for small values of N and compared them with the exact results by the CTM method (Kuniba and Yajima 1988, Pearce and Seaton 1988).

Starting from (5) we have computed SSP by using the formulae of the eigenvectors. Our result is

$$P(s) = \frac{2}{L-1} \prod_{m=1}^{\infty} (1 - x^{2Lm})^3 \left(\frac{1 + x^{2(L-1)m}}{1 - x^{2(L-1)m}} \frac{1 - x^{2m}}{1 + x^{2m}} \right)^2$$
$$\times \sum_{\substack{k:\mathrm{odd} \neq L-1 \\ k \in Z/2(L-1)Z}} \sum_{l \in Z/LZ} \omega_{2(L-1)}^{(s+1-l)k} \frac{E(x^{2l}\omega_{2(L-1)}^{-k}, x^{2L})}{E(-\omega_{2(L-1)}^{k}, x^{2L}) E(-x^{2l}, x^{2L})} \tag{6}$$

where $E(z, q) = \Sigma_{i\in Z}(-z)^i q^{i(i-1)/2}$. Note the symmetries $P(s) = P(L-1-s) = -P(s+L-1)$. The behaviour of $P(s)$ in the vicinity of the critical point ($p \to 0$, i.e. $x \to 1$) is read from the following; for odd k such that $0 < \pm k < L-1$,

$$\sum_{s \in Z/2(L-1)Z} \omega_{2(L-1)}^{-ks} P(s) = \pm \frac{2L(L-1)\log p}{\pi \mathrm{i}} p^{L(L\pm 2k-2)/8(L-1)}(1 + \mathcal{O}(p^{1/2(L-1)})).$$

These quantities are order parameters of this model.

In order to derive (6) we need to compute the norms of the eigenvectors and the matrix elements of σ_N. Let $|f\rangle$ (respectively $\langle g|$) be the eigenvector corresponding to $u_1, \ldots, u_n$ (respectively $v_1, \ldots, v_n$). Adopting Baxter's argument (appendices A and B of Baxter 1973a) for the present situation, we are led to the following expression for $\langle g|f\rangle$:

$$\langle g|f\rangle = (-1)^n \mathrm{e}^{(\pi \mathrm{i}/L)\Sigma_{j=1}^{n}(v_j - u_j)} \sum_{l \in Z/LZ} P_n\left(u \,\middle|\, v;\, l + \frac{L\tau}{2}\right)$$
$$P_n(u|v; \xi) = \sum_{k=1}^{2n-1} \frac{[1][\Sigma_{j=1}^{n}(u_j - v_j) + k - \xi]}{[k-\xi][\Sigma_{j=1}^{n}(u_j - v_j)]} \sum_{r=0}^{n} h^{2r} P_{n,k}^{(r)} \tag{7}$$
$$P_{n,k}^{(r)} = (-1)^{k-1} \frac{[k-2r]}{[1]} \sum_{\substack{|K|=n-r \\ |L|=k-r}} T(u_{K'}|u_K) T(u_K|v_{L'}) T(v_{L'}|v_L) T(v_L|u_{K'}).$$

Let us explain the notation. The parameter h is an electric field similar to the one in Baxter (1973a). It is *on* for $|f\rangle$ and *off* for $\langle g|$. The last sum is over subsets K, L of $\{1, \ldots, n\}$ with the prescribed cardinalities. We abbreviate $\Pi_{i\in K, j\in L} T(u_i, u_j)$ ($T(u, v) = [u-v+1]/[u-v]$) to $T(u_K|u_L)$. The complement of K in $\{1, \ldots, n\}$ is denoted by K'.

We also obtain

$$(1-h^2)\frac{\langle g|\sigma_N|f\rangle}{\langle g|f\rangle} = 1+h^2-2h^2\frac{T(v|\frac{1}{2}-w_N)}{T(u|\frac{1}{2}-w_N)}. \tag{8}$$

The formulae (7) and (8) are valid for arbitrary $\{w_j\}$ and arbitrary solutions $\{u_a\}$ and $\{v_a\}$. To obtain the final result (6) we must further rewrite (7) under the special choice of $\{w_j\}$ (4) and $\{u_a = \alpha(x, h^2) + a\eta\}$, $\{v_a = \beta(x, h^2 = 1) + a\eta\}$. We set $\Delta = \alpha - \beta$.

Under these specialisations, we have

$$\frac{\langle g|f\rangle}{1-h^2} = (-1)^n \frac{nM^n[1]}{[n\Delta]} \sum_{l\in Z/LZ} \frac{[l-n\Delta]_4}{[l]_4} \left\{ \prod_{m=1}^{n-1} (R(\Delta) + h^2R(-\Delta) - r_m) + \mathcal{O}(1-h^2) \right\} \tag{9}$$

where

$$M = \prod_{j=1}^{n-1} \left(e^{-\pi i/L} \frac{[j\eta+1]}{[j\eta]} \right) \qquad R(\Delta) = \frac{[\Delta+1]}{[\Delta]} \prod_{j=1}^{n-1} \frac{[\Delta+j\eta+1]}{[j\eta+1]} \frac{[j\eta]}{[\Delta+j\eta]}$$

$$r_m = -\frac{[2]}{[1]} \sum_{j=1}^{n-1} \omega_n^{jm} \frac{1}{S(j\eta)} \qquad S(u) = \frac{[u+1][u-1]}{[u]^2}.$$

In deriving this formula we admit the following two conjectural identities (A) and (B). Let $x_{r,k}(\Delta) = \Sigma_{|K|=r, |L|=k} S(\Delta + v_K|v_L)/S(v_K)S(v_L)$, where $S(v_K) = \Pi_{a<b, a,b\in K} S((a-b)\eta)$ and $S(\Delta + v_K|v_L) = \Pi_{a\in K, b\in L} S(\Delta + (a-b)\eta)$.

$$x_{r,k+r}(\Delta) = \sum_{0\leqslant i\leqslant r} \binom{n-k-2r+2i}{i} (R(\Delta)R(-\Delta))^i x_{r-i,k+r-i}(1) \qquad (r, k \geqslant 0) \tag{A}$$

$$\sum_{\substack{0\leqslant 2r<k \\ r\in Z}} (k-2r)\frac{[k-2r]}{[1]} x_{r,k-r}(1) = n \sum_{1\leqslant m_1<\ldots<m_{k-1}\leqslant n-1} r_{m_1} \cdots r_{m_{k-1}} \qquad 1 \leqslant k \leqslant n. \tag{B}$$

Finally, taking the limit $n \to \infty$ in (7), (9) we obtain (6).

The authors thank Murray Batchelor, Michael Barber, Rodney Baxter, Barry McCoy, Paul Pearce, Kenji Ueno and Yuji Yamada for useful discussions and generous help at various stages of this work.

References

Baxter R J 1973a *J. Phys. C: Solid State Phys.* **6** L94; *J. Stat. Phys.* **9** 145

—— 1973b *Ann. Phys.* **76** 48

—— 1978 *Phil. Trans. R. Soc.* A **289** 315

—— 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)

Its A R, Izergin A G, Korepin V E and Slavnov N A 1989 Differential equations for quantum correlation functions *Preprint* CMA, ANU, Canberra, Australia

Kirillov A N and Smirnov F A 1988 *Int. J. Mod. Phys.* **3A** 731

Kuniba A and Yajima T 1988 *J. Stat. Phys.* **52** 829

Pearce P A and Batchelor M T 1989 Row transfer matrix spectra of cyclic solid-on-solid models *Preprint* CMA, ANU, Canberra, Australia

Pearce P A and Seaton K A 1988 *Phys. Rev. Lett.* **60** 1347